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Abstract. We study the nonlinear stability of a large class of inhomogeneous steady state solutions to the Hamiltonian Mean Field (HMF) model. Under a specific criterion, we prove the nonlinear stability of steady states which are decreasing functions of the microscopic energy. To achieve this task, we extend to this context the strategy based on generalized rearrangement techniques which was developed recently for the gravitational Vlasov-Poisson equation. Explicit stability inequalities are established and our analysis is able to treat non compactly supported steady states to HMF, which are physically relevant in this context but induces additional difficulties, compared to the Vlasov-Poisson system.

1. Introduction and main result

1.1. **The HMF model.** In this paper, we are interested in the nonlinear stability of a class of inhomogeneous steady state solutions to the Hamiltonian mean-field (HMF) model [18, 1]. The HMF system is a kinetic model describing particles moving on a unit circle interacting via an infinite range attractive cosine potential. This model has been used as a toy-model of the Vlasov-Poisson system in the physical community, for the study of non equilibrium phase transitions [11, 22, 2, 20], of travelling clusters [6, 23] or of relaxation processes [24, 3, 12]. The dynamics of perturbations of inhomogeneous steady states of the HMF model has been investigated in [4, 5] and the formal linear stability of steady states has been studied in [10, 19, 7]. In particular, a simple criterion of linear stability has been derived in [19]. Our aim here is to prove the nonlinear stability of inhomogeneous steady states under the same criterion, by adapting the techniques developed in [15, 16] for the 3D Vlasov-Poisson system. However, we emphasize that the steady state solutions to the Vlasov-Poisson system studied in [15] are compactly supported. Here this assumption is not needed and a weaker assumption is made in the case of the HMF model, as we will see later on. Note finally that the long-time validity of the N-particle approximation for the HMF model has been investigated in [8, 9] and the Landau-damping phenomenon near spatially homogeneous state has been studied recently in [13].

In the HMF model, the distribution function of particles $f(t, \theta, v)$ solves the initial-valued problem

$$\begin{aligned} &\partial_t f + v\partial_\theta f - \partial_\theta \phi_f \partial_v f = 0, \qquad (t, \theta, v) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}, \\ &f(0, \theta, v) = f_{\text{init}}(\theta, v) \geqslant 0, \end{aligned} \tag{1.1}$$

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where $\mathbb{T}$ is the flat torus $[0, 2\pi]$ and where the self-consistent potential ϕ_f associated to a distribution function f is defined by

$$\phi_f(\theta) = -\int_0^{2\pi} \rho_f(\theta') \cos(\theta - \theta') d\theta', \qquad \rho_f(\theta) = \int_{\mathbb{R}} f(\theta, v) dv. \tag{1.2}$$

The so-called magnetization is the two-dimensional vector defined by

$$M_f = \int_0^{2\pi} \rho_f(\theta) u(\theta) d\theta, \qquad \text{with} \quad u(\theta) = (\cos\theta, \sin\theta)^T \tag{1.3}$$

and we have

$$\phi_f(\theta) = -M_f \cdot u(\theta). \tag{1.4}$$

The Cauchy problem for (1.1) is much simpler than the one for the Vlasov-Poisson system, since the interaction kernel is smooth, and it can be shown that the HMF model is well-posed in the natural energy space. The following quantities are invariant during the evolution:

– the Casimir functions

$$\iint G(f(\theta, v)) d\theta dv \tag{1.5}$$

for any function $G \in \mathcal{C}^1(\mathbb{R}_+)$ such that $G(0) = 0$;

– the nonlinear energy

$$\begin{aligned} \mathcal{H}(f) &= \frac{1}{2} \iint v^2 f(\theta, v) d\theta dv + \frac{1}{2} \int \rho_f(\theta) \phi_f(\theta) d\theta \\ &= \frac{1}{2} \iint v^2 f(\theta, v) d\theta dv - \frac{1}{2} M_f \cdot \int \rho_f(\theta) u(\theta) d\theta \\ &= \frac{1}{2} \iint v^2 f(\theta, v) d\theta dv - \frac{1}{2} |M_f|^2; \end{aligned} \tag{1.6}$$

– the total momentum

$$\iint v f(\theta, v) d\theta dv. \tag{1.7}$$

Moreover, the HMF system enjoys the Galilean invariance, that is, if $f(t, \theta, v)$ is a solution, then so is $f(t, \theta + v_0 t, v + v_0)$, for $v_0 \in \mathbb{R}$.

1.2. **Statement of the main result.** We consider a stationary state of the form

$$f_0(\theta, v) = F(e_0(\theta, v)), \qquad \text{with} \quad e_0(\theta, v) = \frac{v^2}{2} + \phi_0(\theta), \tag{1.8}$$

and where the potential associated to f_0 according to (1.2) takes the form

$$\phi_0(\theta) = -m_0 \cos\theta, \qquad \text{with } m_0 > 0.$$

Here F is a given function satisfying the following assumption.

Assumption 1.1. *The function F is a $\mathcal{C}^0$ function on $\mathbb{R}$ satisfying the following properties. It is a $\mathcal{C}^1$ function on $(-\infty, e_*)$, for some $e_* \in \mathbb{R} \cup \{+\infty\}$, with $F' < 0$ on this interval. We also assume that $F(e) = 0$ for $e \geqslant e_*$ when e_* is finite, and that $\lim_{e \to +\infty} F(e) = 0$ if $e_* = +\infty$. We denote by F^{-1} its inverse function, which is a $\mathcal{C}^1$ function defined from $(0, \sup F)$ onto $(-\infty, e^*)$. The function f_0 given by (1.8) is supposed to belong to the energy space $L^1((1 + |v|^2) d\theta dv)$. Moreover, in the case*

$e_* < +\infty$ and $m_0 = e_*$, we assume further that $\int_{-m_0}^{m_0} \log(m_0 - e) F'(e) de < +\infty$.

Examples. All the following typical examples that can be found in the literature fulfill our Assumption 1.1:

(i) Maxwell-Boltzmann distributions [12], $F(e) = A\exp(-\beta e)$.

(ii) Polytropic distributions with compact support [10], $F(e) = A(e_* - e)_+^{\frac{1}{q-1}}$ with $q > 1$. We used the usual notation $x_+ = \max(0, x)$.

(iii) Polytropic distributions with non compact support [10], $F(e) = A(e_0 + e)_+^{\frac{1}{q-1}}$ with $\frac{1}{3} < q < 1$.

(iv) Lynden-Bell distributions [11], $F(e) = \frac{A}{1+B\exp(\beta e)}$.

Remark 1.2. *Note that Assumption 1.1 implies in particular that $f_0 \in L^\infty$ since*

$$\|f_0\|_{L^\infty} \leqslant F(-m_0).$$

It is also clear that e_* is finite if and only if f_0 is compactly supported. We finally note that we must have $e_* > -m_0$, otherwise $f_0 = 0$ and this contradicts the assumption $m_0 > 0$.

Our aim is to prove the orbital stability of such steady state under the following criterion.

Assumption 1.3 (Nonlinear stability criterion)**.** *We will assume that f_0 satisfies the following criterion*

$$\kappa_0 < 1,$$

with

$$\kappa_0 = \int_0^{2\pi}\int_{-\infty}^{+\infty} \left|F'(e_0(\theta, v))\right| \left(\frac{\int_{\mathcal{D}} (\cos\theta - \cos\theta')(e_0(\theta, v) - \phi_0(\theta'))^{-1/2} d\theta'}{\int_{\mathcal{D}} (e_0(\theta, v) - \phi_0(\theta'))^{-1/2} d\theta'} \right)^2 d\theta dv, \tag{1.9}$$

where

$$\mathcal{D} = \left\{\theta' \in \mathbb{T} \,:\, \phi_0(\theta') < e_0(\theta, v)\right\}.$$

Remark 1.4. *Direct computations show that our criterion $\kappa_0 < 1$ is the same as the one derived in* [19], *that is*

$$0 < 1 + \iint F'(e_0(\theta, v)) \cos^2\theta d\theta dv - \frac{4}{\sqrt{m_0}} \int_{-m_0}^{m_0} K(k(e)) \left(\frac{2E(k(e))}{K(k(e))} - 1\right)^2 F'(e) de$$
$$- \frac{4}{\sqrt{m_0}} \int_{m_0}^{+\infty} \frac{K(1/k(e))}{k(e)} \left(\frac{2k(e)^2 E(1/k(e))}{K(1/k(e))} + 1 - 2k(e)^2\right)^2 F'(e) de,$$

with $k(e) = \left(\frac{e+m_0}{2m_0}\right)^{1/2}$ and where $K(k)$ and $E(k)$ are respectively the complete elliptic integrals of first and second kinds, see e.g. [4].

Before stating our main result, we first recall the usual notion of rearrangement which we adapt here to functions defined on the domain $\mathbb{T} \times \mathbb{R}$. For any nonnegative function $f \in L^1(\mathbb{T} \times \mathbb{R})$, we define its distribution function as

$$\mu_f(s) = |\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : f(\theta, v) > s\}|, \quad \text{for all } s \geqslant 0, \tag{1.10}$$

where $|A|$ denotes the Lebesgue measure of a set A. Note that $\mu_f(0)$ may be infinite, but $\mu_f(s)$ is finite for $s > 0$. Let $f^\sharp$ be the pseudo-inverse of the function μ_f, defined by

$$f^\sharp(s) = \inf\{t \geqslant 0,\, \mu_f(t) \leqslant s\} = \sup\{t \geqslant 0,\, \mu_f(t) > s\}, \quad \text{for all } s \geqslant 0$$

with, in particular, $f^\sharp(0) = \|f\|_{L^\infty} \in \mathbb{R} \cup \{+\infty\}$ and $f^\sharp(+\infty) = 0$. It is well known that μ_f is right-continuous and that for all $s \geqslant 0$, $t \geqslant 0$,

$$f^\sharp(s) > t \quad \Longleftrightarrow \quad \mu_f(t) > s. \tag{1.11}$$

Next, we define the rearrangement f^* of f by

$$f^*(\theta, v) = f^\sharp\left(\left|B(0, \sqrt{\theta^2 + v^2}) \cap \mathbb{T} \times \mathbb{R}\right|\right),$$

where $B(0, R)$ denotes the open ball in $\mathbb{R}^2$ centered at 0 with radius R.

Our main result is the following theorem.

Theorem 1.5. *Let f_0 be a steady state of the form* (1.8) *satisfying Assumptions 1.1 and 1.3. There exists $\delta > 0$ such that, for all $f \in L^1\left((1 + |v|^2)d\theta dv\right)$ satisfying $|M_f - M_{f_0(\cdot - \theta_f)}| < \delta$, we have*

$$\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 \leqslant C\Bigg(\mathcal{H}(f) - \mathcal{H}(f_0) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1} \\ + C\int_0^{+\infty} s^2 \left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds + C\int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f^*, f_0^*}(s) ds\Bigg), \tag{1.12}$$

where $\beta_{f^*, f_0^*}(s) = |\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : f^*(\theta, v) \leqslant s < f_0^*(\theta, v)\}|$, for all $s \geqslant 0$, and where C is a positive constant depending only on f_0. The parameter θ_f is defined by $M_f = |M_f|(\cos\theta_f, \sin\theta_f)^T$, where M_f is given by (1.3)*. In particular, if f_0 is a compactly supported steady state, then* (1.12) *can be replaced by*

$$\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 \leqslant C\left(\mathcal{H}(f) - \mathcal{H}(f_0) + C(1 + \|f\|_{L^1})\|f^* - f_0^*\|_{L^1}\right). \tag{1.13}$$

The proof of this theorem is given in Section 5 and uses several steps which are developed in the following sections. In Section 2, we introduce the generalized rearrangements with respect to the microscopic energy, which enable to define a reduced energy function depending on the magnetization vector only. In Section 3 we show that, under the stability criterion $\kappa_0 < 1$, the magnetization of the steady state is a strict local minimizer of this reduced energy function and, in Section 4, we use a result in [14] to establish a functional inequality that enables the control of $f - f_0$. We finally end the proof of Theorem 1.5 in section 5.

1.3. **Proof of the orbital stability of f_0.** In this subsection, we show how to derive a stability result for the HMF model directly from our main Theorem 1.5.

Corollary 1.6. *Let f_0 be a steady state of the form* (1.8) *satisfying Assumptions 1.1 and 1.3. Then f_0 is orbitally stable in the energy space, i.e., for all $\varepsilon > 0$, there exists $\eta > 0$ such that the following holds. For all solution $f(t)$ to the HMF model with initial data f_{init}, that preserves the mass and the energy, we have: if $\|(1+v^2)(f_{init}-f_0)\|_{L^1} \leqslant \eta$, then $\|(1+v^2)(f(\cdot-\theta_f)-f_0)\|_{L^1} \leqslant \varepsilon$, where θ_f is defined by $M_f = |M_f|(\cos\theta_f, \sin\theta_f)^T$ and M_f is given by* (1.3).

Proof. We distinguish the two cases: $e_* < +\infty$ and $e_* = +\infty$.

Case 1: $e_* < +\infty$. In this case f_0 is compactly supported and we can apply (1.13), that is we have

$$\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 \leqslant C\left(\mathcal{H}(f) - \mathcal{H}(f_0) + C(\,1 + \|f\|_{L^1})\|f^* - f_0^*\|\right) \tag{1.14}$$

for all f satisfying $|M_f - M_{f_0(\cdot-\theta_f)}| < \delta$. Let $f_{init} \in L^1\left((1+|v|^2)dvd\theta\right)$ be any initial data for the HMF equation (1.1) such that

$$\|(1+|v|^2)(f_{init} - f_0)\|_{L^1} < \eta,$$

where $0 < \eta < \min(1, \delta/2)$ will be made precise later on. This implies in particular that

$$|M_{f_{init}} - M_{f_0}| \leqslant \|f_{init} - f_0\|_{L^1} < \eta < \delta/2, \tag{1.15}$$

and then

$$\begin{aligned}|\mathcal{H}(f_{init}) - \mathcal{H}(f_0)| &= \left|\iint \left(\frac{v^2}{2} + \phi_0(\theta)\right)(f_{init} - f_0)d\theta dv - \frac{1}{2}\left|M_{f_{init}} - M_{f_0}\right|^2\right| \\ &\leqslant (m_0+1)\eta.\end{aligned}$$

Now the contractivity property of the rearrangement implies that $\|f_{init}^* - f_0^*\|_{L^1} \leqslant \|f_{init} - f_0\|_{L^1} < \eta$ and then

$$\begin{aligned}|\mathcal{H}(f_{init}) - \mathcal{H}(f_0)| &+ C(\,1 + \|f_{init}\|_{L^1})\|f_{init}^* - f_0^*\| \\ &\leqslant \left[m_0 + 1 + C\left(2 + \|f_0\|_{L^1}\right)\right]\eta.\end{aligned}$$

We then choose η such that

$$\eta < \min\left(1, \delta/2, \left[m_0 + 1 + C\left(2 + \|f_0\|_{L^1}\right)\right]^{-1}\delta^2/(4C)\right). \tag{1.16}$$

Let now $f(t)$ be a solution to the HMF model with initial data f_{init}. From the conservation properties of this model, to wit $\mathcal{H}(f(t)) = \mathcal{H}(f_{init})$ and $f(t)^* = f_{init}^*$, and from (1.14) we then get

$$\begin{aligned}\|f(t) - f_0(\cdot - \theta_{f(t)})\|_{L^1}^2 &\leqslant C\left(\mathcal{H}(f(t)) - \mathcal{H}(f_0) + C(\,1 + \|f(t)\|_{L^1})\|f(t)^* - f_0^*\|\right) \\ &< \delta^2/4,\end{aligned}$$

as long as $|M_{f(t)} - M_{f_0(\cdot-\theta_{f(t)})}| < \delta$. In fact we shall prove that we have

$$|M_{f(t)} - M_{f_0(\cdot-\theta_{f(t)})}| < \delta, \quad \forall t \geqslant 0. \tag{1.17}$$

Indeed, at $t = 0$ we have $|M_{f(0)} - M_{f_0(\cdot - \theta_{f(0)})}| < \delta/2$ by assumption on f_{init} (see (1.15)). If at some time t we have $|M_{f(t)} - M_{f_0(\cdot - \theta_{f(t)})}| \geqslant \delta$, then by continuity in time there is some time t_0 such that $|M_{f(t_0)} - M_{f_0(\cdot - \theta_{f(t_0)})}| = 2\delta/3 < \delta$. We thus get

$$\|f(t_0) - f_0(\cdot - \theta_{f(t_0)})\|_{L^1} < \delta/2.$$

But this implies

$$2\delta/3 = |M_{f(t_0)} - M_{f_0(\cdot - \theta_{f(t_0)})}| \leqslant \|f(t_0) - f_0(\cdot - \theta_{f(t_0)})\|_{L^1} < \delta/2,$$

which is a contradiction, and claim (1.17) is proved. We conclude from Theorem 1.5 that

$$\|f(t) - f_0(\cdot - \theta_{f(t)})\|_{L^1}^2 \leqslant C\left(|\mathcal{H}(f_{init}) - \mathcal{H}(f_0)| + C(1 + \|f_{init}\|_{L^1})\|f_{init}^* - f_0^*\|_{L^1}\right), \tag{1.18}$$

for all $t \geqslant 0$. The orbital stability of the solution $f(t)$ is then proved in the L^1 norm in a quantitative way, since the right-hand side of (1.18) goes to zero as $\|(1 + v^2)(f_{init} - f_0)\|_{L^1}$ goes to zero, as a consequence of the usual contractivity property of the rearrangement $\|f_{init}^* - f_0^*\|_{L^1} \leqslant \|f_{init} - f_0\|_{L^1}$. It remains to prove this stability in the whole energy norm. We argue by contradiction. Assume that there exists $\varepsilon > 0$ and a sequence f_{init}^n such that $\|(1 + v^2)(f_{init}^n - f_0)\|_{L^1} \to 0$ as $n \to +\infty$ and, for some $t^n > 0$, we have

$$\inf_{\tilde{\theta} \in [0, 2\pi]} \|v^2(g^n(\cdot - \tilde{\theta}) - f_0)\|_{L^1} > \varepsilon,$$

where $g^n = f^n(t^n)$ and f^n is a solution of the HMF model associated with the initial data f_{init}^n. We have already shown the L^1 stability, which means that we have $\|g^n(\cdot - \theta_{g_n}) - f_0\|_{L^1} \to 0$ as $n \to +\infty$. In particular, up to a subsequence, we have $v^2 g^n(\cdot - \theta_{g_n}) \to v^2 f_0$ almost everywhere as $n \to \infty$. Now from the Brézis-Lieb lemma, we have

$$\|v^2 g^n(\cdot - \theta_{g_n}) - v^2 f_0\|_{L^1} - \|v^2 g^n(\cdot - \theta_{g_n})\|_{L^1} + \|v^2 f_0\|_{L^1} \to 0 \tag{1.19}$$

as $n \to +\infty$. From the conservation of the energy and the convergence $\|(1+v^2)(f_{init}^n - f_0)\|_{L^1} \to 0$, we have

$$\mathcal{H}(g^n(\cdot - \theta_{g^n})) = \mathcal{H}(g^n) = \mathcal{H}(f_{init}^n) \to \mathcal{H}(f_0). \tag{1.20}$$

Note that we have used the convergence of the magnetization vector

$$\left|M_{f_{init}^n} - M_{f_0}\right| \leqslant \|f_{init}^n - f_0\|_{L^1} \to 0.$$

We apply this inequality to g^n

$$\left|M_{g^n(\cdot - \theta_{g^n})} - M_{f_0}\right| \leqslant \|g^n(\cdot - \theta_{g^n}) - f_0\|_{L^1} \to 0,$$

and obtain from (1.20)

$$\|v^2 g^n(\cdot - \theta_{g^n})\|_{L^1} - \|v^2 f_0\|_{L^1} \to 0.$$

Using (1.19), this implies that

$$\|v^2 g^n(\cdot - \theta_{g^n}) - v^2 f_0\|_{L^1} \to 0,$$

and yields a contradiction.

Case 2: $e_* = +\infty$. In this case f_0 is not compactly supported and we shall use inequality (1.12) of Theorem 1.5. The quantity $\mu_{f_0}(s)$ involved in (1.12) is no longer bounded and presents a singularity at $s = 0$. Therefore we shall need to prove the following claim: if $\|f^n_{init} - f_0\|_{L^1} \to 0$ then

$$\int_0^{+\infty} s^2 \left(f_0^\sharp(s) - f^{n\sharp}_{init}(s)\right)_+ ds \to 0 \quad \text{and} \quad \int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f^{n*}_{init}, f_0^*}(s) ds \to 0, \quad (1.21)$$

as $n \to +\infty$ up to the extraction of a subsequence. Once this claim is proved, the rest of the stability proof is exactly the same as in the case of a compactly supported steady state f_0. Let us then prove claim (1.21). We start by proving the first limit of this claim. Assume that $\|f^n_{init} - f_0\|_{L^1} \to 0$. Since $\|f^{n\sharp}_{init} - f_0^\sharp\|_{L^1(\mathbb{R}^+)} \leqslant \|f^n_{init} - f_0\|_{L^1} \to 0$, we deduce that $s^2 \left(f_0^\sharp(s) - f^{n\sharp}_{init}(s)\right)_+ \to 0$ as $n \to +\infty$ for almost every $s \geqslant 0$, up to an extraction of a subsequence. But we have

$$s^2 \left(f_0^\sharp(s) - f^{n\sharp}_{init}(s)\right)_+ \leqslant s^2 f_0^\sharp(s), \quad \text{and} \quad \int_0^{+\infty} s^2 f_0^\sharp(s) < +\infty,$$

(see (3.4) for the second inequality). Therefore, by dominated convergence we can pass to the limit inside the integral and get the first convergence in claim (1.21). Now we prove the second limit of claim (1.21). Assume again that $\|f^n_{init} - f_0\|_{L^1} \to 0$, then

$$\int_0^{+\infty} \beta_{f^{n*}_{init}, f_0^*}(s) ds = \iint (f_0^* - f^{n*}_{init})_+ d\theta dv \leqslant \|f^{n*}_{init} - f_0^*\|_{L^1} \leqslant \|f^n_{init} - f_0\|_{L^1} \to 0$$

as $n \to +\infty$. This means that $\beta_{f^{n*}_{init}, f_0^*}(s) \to 0$ for almost every $s \geqslant 0$, up to an extraction of a subsquence. We then conclude that the quantity $\mu_{f_0}(s)^2 \beta_{f^{n*}_{init}, f_0^*}(s)$ arising in (1.21) converges to 0 for almost every $s \geqslant 0$ (up to an extraction). Therefore, to end the proof of the second limit in claim (1.21), it is sufficient to dominate this quantity by an L^1 function in $s \in (0, \|f_0\|_{L^\infty})$ uniformly in n. To this purpose, we observe that $\beta_{f^{n*}_{init}, f_0^*}(s) \leqslant \mu_{f_0}(s)$ and then

$$\mu_{f_0}(s)^2 \beta_{f^{n*}_{init}, f_0^*}(s) \leqslant \mu_{f_0}(s)^3, \qquad \forall\ s > 0.$$

To prove that the rhs of this inequality is integrable on $\mathbb{R}_+$, we write

$$\int_0^{+\infty} s^2 f_0^\sharp(s) ds = \int_0^{+\infty} s^2 \left(\int_0^{f_0^\sharp(s)} dt \right) ds = \int_0^{+\infty} \left(\int_{f_0^\sharp(s) > t} s^2 ds \right) dt$$

and using (1.11) we get

$$\int_0^{+\infty} s^2 f_0^\sharp(s) ds = \int_0^{+\infty} \left(\int_{0 \leqslant s < \mu_{f_0}(t)} s^2 ds \right) dt = \frac{1}{3} \int_0^{+\infty} \mu_{f_0}(t)^3 dt.$$

Since from (3.4) we have $\int_0^{+\infty} s^2 f_0^\sharp(s) ds < +\infty$, the proof of claim (1.21) is complete. This proves the orbital L^1 stability. To get the stability in the energy space, we proceed as in the case $e_* < +\infty$. This ends the proof of the orbital stability in all cases.

□

2. **The reduced energy functional**

The aim of this section is to introduce a reduced energy functional $\mathcal{J}(|M_f|)$ which depends only on the modulus of the magnetization and which is such that $\mathcal{J}(|M_f|) - \mathcal{J}(m_0)$ (recall that $M_{f_0} = (m_0, 0)^T$ with $m_0 \geqslant 0$) is controlled by the relative nonlinear energy $\mathcal{H}(f) - \mathcal{H}(f_0)$, up to conserved quantities.

2.1. **Generalized rearrangements with respect to the microscopic energy.**

Our purpose now is to define a generalized symmetric nonincreasing rearrangement with respect to the microscopic energy $e = \frac{v^2}{2} + \phi(\theta)$, where the potential ϕ is a given $\mathcal{C}^\infty$ function on $\mathbb{T}$. We introduce the quantity

$$a_\phi(e) = \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : \ \frac{v^2}{2} + \phi(\theta) < e\right\}\right|, \quad \text{for all } e \in \mathbb{R}. \tag{2.1}$$

It has the explicit expression

$$a_\phi(e) = 2\sqrt{2} \int_0^{2\pi} \sqrt{(e - \phi(\theta))_+}\, d\theta.$$

It is readily seen that a_ϕ is continuous on $\mathbb{R}$, vanishes on $(-\infty, \min \phi]$ and is strictly increasing from $[\min \phi, +\infty)$ to $[0, +\infty)$. This enables to define its inverse a_ϕ^{-1} on $[0, +\infty)$. Note that, for all $e \in \mathbb{R}$,

$$4\pi\sqrt{2}(e - \max \phi)_+^{1/2} \leqslant a_\phi(e) \leqslant 4\pi\sqrt{2}(e - \min \phi)_+^{1/2}, \tag{2.2}$$

which implies, for all $s \in \mathbb{R}_+$,

$$\frac{s^2}{32\pi^2} + \min \phi \leqslant a_\phi^{-1}(s) \leqslant \frac{s^2}{32\pi^2} + \max \phi. \tag{2.3}$$

We now introduce the generalized rearrangement with respect to the microscopic energy.

Lemma 2.1. *Let $\phi \in \mathcal{C}^\infty(\mathbb{T})$ and let a_ϕ be the function defined by* (2.1)*. Let $f \in L^1(\mathbb{T} \times \mathbb{R})$, nonnegative. Then the function*

$$f^{*\phi}(\theta, v) = f^\sharp\left(a_\phi\left(\frac{v^2}{2} + \phi(\theta)\right)\right), \qquad (\theta, v) \in \mathbb{T} \times \mathbb{R}$$

is equimeasurable to f, that is $\mu_{f^{*\phi}} = \mu_f$, where μ_f is defined by (1.10)*. In the sequel, the function $f^{*\phi}$ is called (decreasing) rearrangement with respect to the microscopic energy $\frac{v^2}{2} + \phi(\theta)$.*

Proof. Recall that from the right continuity of μ_f, we have (1.11), for all $s \geqslant 0$, $t \geqslant 0$. Therefore, for all $t \geqslant 0$,

$$\begin{aligned}
\mu_{f^{*\phi}}(t) &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : \quad f^\sharp\left(a_\phi\left(\frac{v^2}{2} + \phi(\theta)\right)\right) > t\right\}\right| \\
&= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : \quad a_\phi\left(\frac{v^2}{2} + \phi(\theta)\right) < \mu_f(t)\right\}\right| \\
&= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} : \quad \frac{v^2}{2} + \phi(\theta) < a_\phi^{-1}(\mu_f(t))\right\}\right| \\
&= a_\phi\left(a_\phi^{-1}(\mu_f(t))\right) = \mu_f(t).
\end{aligned}$$

□

Finally, we state a technical lemma dealing with the case of potentials which have the special form of potentials of the HMF model. For $e \in \mathbb{R}$, $m \in \mathbb{R}_+^*$ and $\phi(\theta) = -m\cos\theta$ we denote

$$\alpha_m(e) = a_\phi(e) = 2\sqrt{2}\int_0^{2\pi} \sqrt{(e + m\cos\theta)_+}\, d\theta = \sqrt{m}\,\alpha_1\left(\frac{e}{m}\right)$$

and introduce the angle

$$\theta_m(e) = \begin{cases} 0, & \text{if } e \leqslant -m, \\ \arccos\,(-e/m) \in (0,\pi), & \text{if } -m < e < m, \\ \pi, & \text{if } e \geqslant m. \end{cases} \tag{2.4}$$

The function $\alpha_1(e)$ and its derivative $\alpha_1'(e)$ are represented on Figure 1. The proof

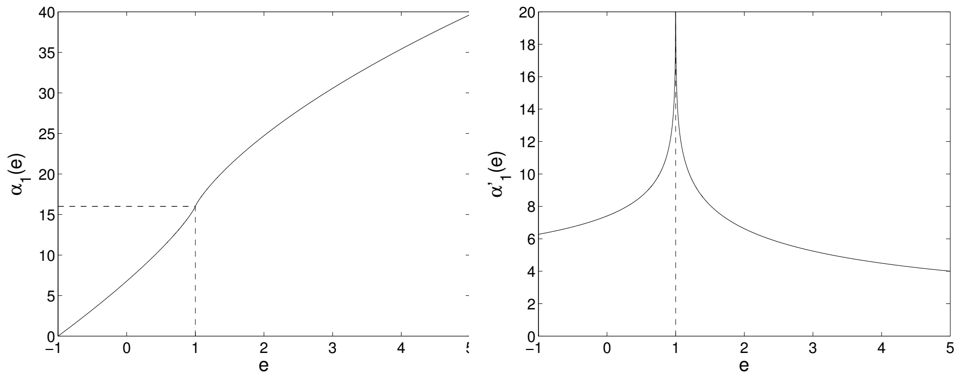

FIGURE 1. Function α_1 (left) and its derivative α_1' (right).

of the following lemma is deffered to the Appendix.

Lemma 2.2 (Properties of the function α_1)**.** *Let*

$$\alpha_1(e) = 4\sqrt{2}\int_0^{\theta_1(e)} (e + \cos\theta)^{1/2}\, d\theta \qquad \text{for } e \in \mathbb{R}. \tag{2.5}$$

This function satisfies the following properties:
(i) α_1 is a continuous nondecreasing function from $\mathbb{R}$ to $\mathbb{R}_+$ and $\alpha_1(e) = 0$ for $e \leqslant -1$.
(ii) α_1 is a strictly increasing and strictly convex $\mathcal{C}^1$ function on $[-1, 1)$. Its derivative for $e \in (-1, 1)$ is given by

$$\alpha_1'(e) = 2\sqrt{2}\int_0^{\theta_1(e)} (e + \cos\theta)^{-1/2}\, d\theta. \tag{2.6}$$

and its right-derivative at $e = -1$ is equal to 2π.
(iii) α_1 is a strictly increasing and strictly concave $\mathcal{C}^1$ function on $(1, +\infty)$. Its derivative for $e \in (1, \infty)$ is still given by (2.6) and we have

$$\alpha_1(e) \sim 4\pi\sqrt{2e}, \qquad \alpha_1'(e) \sim 2\pi\sqrt{2/e} \quad \text{as} \quad e \to +\infty.$$

(iv) We have $\alpha_1(1) = 16$ and

$$\alpha_1'(e) \sim -2\log|e-1| \quad \text{as} \quad e \to 1. \tag{2.7}$$

(v) The inverse α_1^{-1} of the function $\alpha_1 : [-1,+\infty) \mapsto [0,+\infty)$ is a strictly increasing $\mathcal{C}^1$ function, defined on $[0,+\infty)$, satisfying

$$(\alpha_1^{-1})'(s) = \frac{1}{\alpha_1' \circ \alpha_1^{-1}(s)} \quad \textit{for} \quad s \in \mathbb{R}_+ \backslash \{0, 16\},$$

with α_1' given by (2.6), and

$$(\alpha_1^{-1})'(0) = \frac{1}{2\pi}, \qquad (\alpha_1^{-1})'(16) = 0.$$

2.2. **Reduction to a functional of the magnetization vector.** In this subsection, we prove the following result.

Proposition 2.3. *For all $f \in L^1((1+|v|^2)d\theta dv)$, we have*

$$\begin{aligned}\mathcal{J}(|M_f|) - \mathcal{J}(|M_{f_0}|) \leqslant \mathcal{H}(f) - \mathcal{H}(f_0) + 3\|f\|_{L^1}\|f^* - f_0^*\|_{L^1} \\ + \frac{1}{32\pi^2}\int_0^{+\infty} s^2 \left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds\end{aligned} \tag{2.8}$$

where, for all $m \in \mathbb{R}_+$,

$$\mathcal{J}(m) = \frac{m^2}{2} + \int_{-\infty}^{+\infty}\int_0^{2\pi} \left(\frac{v^2}{2} + \phi\right) f_0^{*\phi} d\theta dv \tag{2.9}$$

with $\phi(\theta) = -m\cos\theta$.

Proof. Writing the difference $\mathcal{H}(f) - \mathcal{H}(f_0)$ between the nonlinear energies as

$$\begin{aligned}\mathcal{H}(f) - \mathcal{H}(f_0) &= \iint \left(\frac{v^2}{2} + \phi_f\right)(f - f_0)\, d\theta dv - \frac{1}{2}\left(|M_f|^2 - |M_{f_0}|^2\right) - \iint \phi_f (f - f_0) d\theta dv \\ &= \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f - f^{*\phi_f}\right) d\theta dv + \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f^{*\phi_f} - f_0^{*\phi_f}\right) d\theta dv \\ &\quad + \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f_0^{*\phi_f} - f_0\right) d\theta dv + \frac{1}{2}|M_f - M_{f_0}|^2 \\ &= I_1 + I_2 + I_3 + \frac{1}{2}|M_f - M_{f_0}|^2,\end{aligned}$$

we organize the proof in three steps.

Step 1: Identification of $\mathcal{J}(|M_f|) - \mathcal{J}(|M_{f_0}|)$.

Let us first prove that

$$f_0 = f_0^{*\phi_0}, \tag{2.10}$$

which amounts to proving that

$$F(e) = f_0^\sharp \circ a_{\phi_0}(e), \qquad \forall e \geqslant \min \phi_0. \tag{2.11}$$

Recall that a_{ϕ_0} is invertible from $[\min\phi_0, +\infty)$ to $[0,+\infty)$ and denote $G = F \circ a_{\phi_0}^{-1}$ on $[0,+\infty)$. Recall also that F is assumed to be continuously decreasing. Hence, so is the function G and then it is standard that $G^\sharp = G$, see for instance [21]. Now, for all $t \geqslant 0$,

$$\begin{aligned}\mu_{f_0}(t) &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} \,:\, F\left(\frac{v^2}{2} + \phi_0(\theta)\right) > t\right\}\right| \\ &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} \,:\, G \circ a_{\phi_0}\left(\frac{v^2}{2} + \phi_0(\theta)\right) > t\right\}\right| \\ &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} \,:\, G^\sharp \circ a_{\phi_0}\left(\frac{v^2}{2} + \phi_0(\theta)\right) > t\right\}\right|.\end{aligned}$$

Hence, applying the (1.11) to the function

$$\mu_G(s) = |\{t \geqslant 0 : G(t) > s\}|, \quad \text{for all } s \geqslant 0,$$

and to its pseudo-inverse $G^\sharp$, we get

$$\begin{aligned}\mu_{f_0}(t) &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} \,:\, a_{\phi_0}\left(\frac{v^2}{2} + \phi_0(\theta)\right) < \mu_G(t)\right\}\right| \\ &= \left|\left\{(\theta, v) \in \mathbb{T} \times \mathbb{R} \,:\, \frac{v^2}{2} + \phi_0(\theta) < a_{\phi_0}^{-1}(\mu_G(t))\right\}\right| \\ &= a_{\phi_0} \circ a_{\phi_0}^{-1}(\mu_G(t)) = \mu_G(t).\end{aligned}$$

From this, we deduce that $f_0^\sharp = G^\sharp = G$, which gives (2.11) and ends the proof of (2.10). Consequently,

$$\begin{aligned}&I_3 + \frac{1}{2}|M_f - M_{f_0}|^2 \\ &= \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f_0^{*\phi_f} - f_0\right) d\theta dv + \frac{1}{2}|M_f - M_{f_0}|^2 \\ &= \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f_0^{*\phi_f} - f_0^{*\phi_0}\right) d\theta dv + \frac{1}{2}|M_f - M_{f_0}|^2 \\ &= \iint \left(\frac{v^2}{2} + \phi_f\right) f_0^{*\phi_f} d\theta dv - \iint \left(\frac{v^2}{2} + \phi_0\right) f_0^{*\phi_0} d\theta dv \\ &\quad + \int (\phi_0 - \phi_f)\, \rho_{f_0}\, d\theta + \frac{1}{2}|M_f - M_{f_0}|^2 \\ &= \iint \left(\frac{v^2}{2} + \phi_f\right) f_0^{*\phi_f} d\theta dv + \frac{1}{2}|M_f|^2 - \iint \left(\frac{v^2}{2} + \phi_0\right) f_0^{*\phi_0} d\theta dv - \frac{1}{2}|M_{f_0}|^2.\end{aligned}$$

We observe now that ϕ_f can be written as $\phi_f(\theta) = -|M_f| \cos(\theta - \theta_M)$ for some $\theta_M \in \mathbb{T}$. Hence, by periodicity, we have

$$\iint \left(\frac{v^2}{2} + \phi_f\right) f_0^{*\phi_f} d\theta dv + \frac{1}{2}|M_f|^2 = \mathcal{J}(|M_f|),$$

where $\mathcal{J}$ is defined by (2.9), and the same holds for ϕ_0. We thus have

$$I_3 + \frac{1}{2}|M_f - M_{f_0}|^2 = \mathcal{J}(|M_f|) - \mathcal{J}(|M_{f_0}|).$$

Step 2: Positivity of I_1.

We have, using Fubini,

$$\begin{aligned} I_1 &= \iint \left(\frac{v^2}{2}+\phi_f\right)\left(f-f^{*\phi_f}\right)d\theta dv \\ &= \iint \left(\frac{v^2}{2}+\phi_f\right)\left(\int_0^f dt - \int_0^{f^{*\phi_f}} dt\right)d\theta dv \\ &= \int_0^{+\infty}\left(\iint_{f>t}\left(\frac{v^2}{2}+\phi_f\right)d\theta dv - \iint_{f^{*\phi_f}>t}\left(\frac{v^2}{2}+\phi_f\right)d\theta dv\right)dt \\ &= \int_0^{+\infty}\left(\iint_{A(t)}\left(\frac{v^2}{2}+\phi_f\right)d\theta dv - \iint_{B(t)}\left(\frac{v^2}{2}+\phi_f\right)d\theta dv\right)dt \end{aligned}$$

where, for all $t \geqslant 0$, we have denoted

$$\begin{aligned} A(t) &= \left\{(\theta,v)\in\mathbb{T}\times\mathbb{R} \,:\, f^{*\phi_f}(\theta,v)\leqslant t < f(\theta,v)\right\}, \\ B(t) &= \left\{(\theta,v)\in\mathbb{T}\times\mathbb{R} \,:\, f(\theta,v)\leqslant t < f^{*\phi_f}(\theta,v)\right\}. \end{aligned}$$

Since $f^{*\phi_f}$ is a decreasing function of $\frac{v^2}{2}+\phi_f$, we clearly have

$$\forall(\theta,v)\in A(t),\quad \forall(\theta',v')\in B(t),\qquad \frac{v^2}{2}+\phi_f(\theta) > \frac{v'^2}{2}+\phi_f(\theta').$$

Moreover, from the equimeasurability of f and $f^{*\phi_f}$, we have $|A(t)| = |B(t)|$. Consequently, we obtain $I_1 \geqslant 0$.

Step 3: Control of $|I_2|$ by $\|f^* - f_0^*\|_{L^1}$.

Let us first state an elementary result.

Lemma 2.4. *Let $\phi(\theta) = -m\cos(\theta-\theta_0)$ for $(m,\theta_0)\in\mathbb{R}_+\times\mathbb{T}$. Then, for all $f \in L^1_+(\mathbb{T}\times\mathbb{R})$, we have*

$$\iint\left(\frac{v^2}{2}+\phi(\theta)\right)f^{*\phi}(\theta,v)d\theta dv = \int_0^{+\infty} f^\sharp(s)a_\phi^{-1}(s)ds. \tag{2.12}$$

Proof of Lemma 2.4. By a first change of variable with respect to v: $e = \frac{v^2}{2}+\phi(\theta)$, we get

$$\begin{aligned} \iint\left(\frac{v^2}{2}+\phi\right)f^{*\phi}d\theta dv &= \sqrt{2}\int_0^{2\pi}\int_{\phi(\theta)}^{+\infty} f^\sharp\circ a_\phi(e)e(e-\phi(\theta))^{-1/2}ded\theta \\ &= \sqrt{2}\int_{-m}^{+\infty}\int_{\phi(\theta)<e} f^\sharp\circ a_\phi(e)e(e-\phi(\theta))^{-1/2}d\theta de. \end{aligned}$$

Now, if $m>0$, we deduce from Lemma 2.2 that $e\mapsto a_\phi(e) = \sqrt{m}\alpha_1(e/m)$ is a strictly increasing $\mathcal{C}^1$ diffeomorphims from $E_m = (-m,m)\cup(m,+\infty)$ onto $\mathbb{R}_+^*$. Moreover, from (2.6), we get

$$a'_\phi(e) = \sqrt{2}\int_{\phi(\theta)<e}(e-\phi(\theta))^{-1/2}d\theta,$$

and

$$\iint \left(\frac{v^2}{2} + \phi\right) f^{*\phi} d\theta dv = \int_{e \in E_m} f^\sharp \circ a_\phi(e) e a'_\phi(e) de,$$

so, performing the change of variable $s = a_\phi(e)$ on E_m, we obtain (2.12). If $m = 0$, we observe that $a_\phi(e) = 4\pi\sqrt{2e}$, $a_\phi^{-1}(s) = \frac{s^2}{32\pi^2}$ and

$$\iint \left(\frac{v^2}{2} + \phi\right) f^{*\phi} d\theta dv = 2\pi\sqrt{2}\int_0^{+\infty} f^\sharp\left(4\pi\sqrt{2e}\right)\sqrt{e}de = \int_0^{+\infty} f^\sharp(s)\frac{s^2}{32\pi^2}ds.$$

The proof of the lemma is complete. □

From (2.12), we deduce

$$\begin{aligned}
I_2 &= \iint \left(\frac{v^2}{2} + \phi_f\right)\left(f^{*\phi_f} - f_0^{*\phi_f}\right) d\theta dv \\
&= \int_0^{+\infty} \left(f^\sharp(s) - f_0^\sharp(s)\right) a_{\phi_f}^{-1}(s) ds \\
&= \int_0^{+\infty} \left(f^\sharp(s) - f_0^\sharp(s)\right)\left(a_{\phi_f}^{-1}(s) - \min\phi_f\right) ds + \min\phi_f \int_0^{+\infty}\left(f^\sharp(s) - f_0^\sharp(s)\right) ds \\
&\geqslant \int_{f^\sharp(s) < f_0^\sharp(s)} \left(a_{\phi_f}^{-1}(s) - \min\phi_f\right)\left(f^\sharp(s) - f_0^\sharp(s)\right) ds - \|\phi_f\|_{L^\infty}\|f^\sharp - f_0^\sharp\|_{L^1} \\
&\geqslant -\int_0^{+\infty} a_{\phi_f}^{-1}(s)\left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds - 2\|\phi_f\|_{L^\infty}\|f^\sharp - f_0^\sharp\|_{L^1}.
\end{aligned}$$

Using (2.3), we deduce that

$$I_2 \geqslant -\frac{1}{32\pi^2}\int_0^{+\infty} s^2\left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds - 3\|\phi_f\|_{L^\infty}\|f^\sharp - f_0^\sharp\|_{L^1}.$$

We now conclude by observing that, for all $\theta \in \mathbb{T}$, we have

$$|\phi_f(\theta)| \leqslant |M_f|\,|u(\theta)| = |M_f| \leqslant \|f\|_{L^1}. \tag{2.13}$$

□

3. Study of the functional $\mathcal{J}$.

In this section, we study the function $\mathcal{J}(m)$ defined for $m \in \mathbb{R}_+$ by (2.9), with $\phi(\theta) = -m\cos\theta$. For $e \in \mathbb{R}$ and $m \in \mathbb{R}_+^*$, we recall that

$$a_\phi(e) = \alpha_m(e) = \sqrt{m}\,\alpha_1\left(\frac{e}{m}\right),$$

where α_1 was defined by (2.5). Clearly, (2.9) and (2.12) yield, for $m > 0$,

$$\begin{aligned}
\mathcal{J}(m) &= \frac{m^2}{2} + \int_{-\infty}^{+\infty}\int_0^{2\pi}\left(\frac{v^2}{2} - m\cos\theta\right) f_0^\sharp \circ \alpha_m\left(\frac{v^2}{2} - m\cos\theta\right) d\theta dv \\
&= \frac{m^2}{2} + m\int_0^{+\infty} f_0^\sharp(s)\,\alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right) ds.
\end{aligned}$$

Proposition 3.1. *The function $\mathcal{J}$ defined by* (2.9) *is a $\mathcal{C}^2$ function on $\mathbb{R}_+$. Denoting $\phi(\theta) = -m\cos\theta$, we have*

$$\mathcal{J}'(m) = m - \iint f_0^{*\phi}(\theta, v)\cos\theta \, d\theta dv \tag{3.1}$$

and

$$\mathcal{J}''(m) = 1 + \iint (f_0^\sharp \circ a_\phi)'(e(\theta, v)) \left(\cos\theta - \frac{\displaystyle\int_0^{\theta_m(e(\theta,v))} \cos\theta'(e(\theta, v) + m\cos\theta')^{-1/2} \, d\theta'}{\displaystyle\int_0^{\theta_m(e(\theta,v))} (e(\theta, v) + m\cos\theta')^{-1/2} \, d\theta'} \right)^2 d\theta dv, \tag{3.2}$$

where $e(\theta, v) = \frac{v^2}{2} + \phi(\theta)$ and θ_m is defined by (2.4).

From this Proposition and from (2.10), it is immediate to deduce the

Corollary 3.2. *Under Assumption 1.3, the magnetization m_0 of the stationary state f_0 is a strict local minimizer of $\mathcal{J}$: one has*

$$\mathcal{J}'(m_0) = 0 \qquad \text{and} \qquad \mathcal{J}''(m_0) = 1 - \kappa_0 > 0.$$

Proof of Proposition 3.1. To differentiate the function $\mathcal{J}(m)$, we denote

$$g(m, s) = f_0^\sharp(s)\, \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right).$$

From Lemma 2.2, g is continuously differentiable with respect to $m \in \mathbb{R}_+^*$, with

$$\frac{\partial g}{\partial m}(m, s) = -\frac{s f_0^\sharp(s)}{2m^{3/2}\, \alpha_1' \circ \alpha_1^{-1}(\frac{s}{\sqrt{m}})}.$$

Moreover, we can also easily deduce from Lemma 2.2 that there exists a constant $C > 0$ such that

$$\sqrt{2 + e}\, \alpha_1'(e) \geqslant C, \qquad \forall e \geqslant -1. \tag{3.3}$$

Let us fix $0 < m_1 < m_2$. We deduce from (3.3) that, for all $(m, s) \in [m_1, m_2] \times \mathbb{R}_+$,

$$\left|\frac{\partial g}{\partial m}(m, s)\right| \lesssim s f_0^\sharp(s) \left(2 + \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right)\right)^{1/2},$$

where $f \lesssim g$ means $f \leqslant Cg$ for some constant C. Next, using (2.3), we obtain

$$\left|\frac{\partial g}{\partial m}(m, s)\right| \lesssim (1 + s^2) f_0^\sharp(s).$$

Now, we claim that

$$\int_0^{+\infty} (1 + s^2) f_0^\sharp(s) ds < +\infty. \tag{3.4}$$

Indeed, we already know that $\int f_0^\sharp(s)ds = \|f_0\|_{L^1} < +\infty$ and, by (2.3) and (2.12),

$$\begin{aligned}\int_0^{+\infty} s^2 f_0^\sharp(s)ds &\lesssim \int_0^{+\infty} \left(1 + a_{\phi_0}^{-1}(s)\right) f_0^\sharp(s)ds \\ &= \iint \left(1 + \frac{v^2}{2} + \phi_0(\theta)\right) f_0^{*\phi_0}(\theta, v)d\theta dv \\ &\lesssim \iint \left(1 + v^2 + \|f_0\|_{L^1}\right) f_0(\theta, v)d\theta dv < +\infty,\end{aligned}$$

where we used (2.10), (2.13) and Assumption 1.1. This proves (3.4) and, by dominated convergence, one can continuously differentiate $\mathcal{J}(m) = \frac{m^2}{2} + m\int_0^{+\infty} g(m,s)ds$ for all $m > 0$:

$$\begin{aligned}\mathcal{J}'(m) &= m + \int_0^{+\infty} g(m,s)ds + m\int_0^{+\infty} \frac{\partial g}{\partial m}(m,s)ds \\ &= m + \int_0^{+\infty} f_0^\sharp(s) \left(\alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right) - \frac{s}{2\sqrt{m}\,\alpha_1' \circ \alpha_1^{-1}(\frac{s}{\sqrt{m}})}\right) ds \\ &= m + \frac{1}{m}\int_0^{+\infty} f_0^\sharp(s) \left(a_\phi^{-1}(s) - \frac{s}{2a_\phi' \circ a_\phi^{-1}(s)}\right) ds \\ &= m + \frac{1}{m}\int_{-m}^{+\infty} f_0^\sharp \circ a_\phi(e) \left(e a_\phi'(e) - \frac{1}{2}a_\phi(e)\right) de.\end{aligned}$$

We now introduce the function

$$\beta_1(e) = 2\sqrt{2}\int_0^{2\pi} \cos\theta\sqrt{(e + \cos\theta)_+}\, d\theta \qquad \text{for } e \in \mathbb{R}$$

and denote

$$b_\phi(e) = 2\sqrt{2}\int_0^{2\pi} \cos\theta\sqrt{(e + m\cos\theta)_+}\, d\theta = \sqrt{m}\,\beta_1\left(\frac{e}{m}\right). \tag{3.5}$$

Let us list a few properties of this function b_ϕ. By adapting the proof of Lemma 2.2 developed in the Appendix, it is readily seen that b_ϕ is a continuous function on $\mathbb{R}$, vanishing for $e \leqslant -m$, continuously differentiable on $[-m, m) \cup (m, +\infty)$ with

$$b_\phi'(e) = 2\sqrt{2}\int_0^{\theta_m(e)} \cos\theta(e + m\cos\theta)^{-1/2}\, d\theta.$$

Moreover, we have

$$b_\phi(e) = 4\sqrt{2}\int_0^{\pi/2} \cos\theta\left(\sqrt{(e + m\cos\theta)_+} - \sqrt{(e - m\cos\theta)_+}\right) d\theta$$

which implies that $b_\phi(e)$ is always positive for $e > -m$, $m > 0$. For $e > m$ we then have

$$b_\phi(e) = 8m\sqrt{2}\int_0^{\pi/2} \frac{(\cos\theta)^2}{\sqrt{e + m\cos\theta} + \sqrt{(e - m\cos\theta)_+}}\, d\theta, \tag{3.6}$$

$$b_\phi(e) \sim \frac{\pi m\sqrt{2}}{\sqrt{e}} \qquad \text{as } e \to +\infty. \tag{3.7}$$

Similarly, for $e > m$ we have

$$b'_\phi(e) = -4m\sqrt{2}\int_0^{\pi/2} \frac{(\cos\theta)^2}{\sqrt{e+m\cos\theta}\sqrt{e-m\cos\theta}(\sqrt{e+m\cos\theta}+\sqrt{e-m\cos\theta}} \, d\theta,$$

thus

$$b'_\phi(e) \sim -\frac{\pi m}{e\sqrt{2e}} \qquad \text{as } e \to +\infty. \tag{3.8}$$

Now we observe that

$$ea'_\phi(e) + mb'_\phi(e) = \frac{1}{2}a_\phi(e). \tag{3.9}$$

Hence, for $m > 0$, we have

$$\begin{aligned} \mathcal{J}'(m) &= m - \int_{-m}^{+\infty} f_0^\sharp \circ a_\phi(e) b'_\phi(e)\, de \qquad (3.10)\\ &= m - 2\sqrt{2}\int_{-m}^{+\infty}\int_0^{\theta_m(e)} f_0^\sharp \circ a_\phi(e)\frac{\cos\theta}{\sqrt{e+m\cos\theta}}\, d\theta de. \end{aligned}$$

By passing to the limit in this formula, we also get that $\mathcal{J}$ is differentiable at $m = 0$, with $\mathcal{J}'(0) = 0$. Finally, coming back to the variables (θ, v), we obtain (3.1).

In order to compute the second derivative of $\mathcal{J}$, let us transform this expression into a more suitable one, using an integration by parts in e. We denote $\widetilde{e}_* = a_\phi^{-1} \circ a_{\phi_0}(e_*)$, where e_* is defined in Assumption 1.1. By (2.10), we have $f_0^\sharp \circ a_\phi = F \circ a_{\phi_0}^{-1} \circ a_\phi$, this function being continuous on $[-m, +\infty)$, of class $\mathcal{C}^1$ on $[-m, +\infty)\backslash\{m, \widetilde{e}_*\}$, nonincreasing, and vanishes on $[\widetilde{e}_*, +\infty)$. Therefore, in the case $e_* < +\infty$, one can directly integrate by parts to obtain

$$\int_{-m}^{+\infty} f_0^\sharp \circ a_\phi(e) b'_\phi(e)\, de = -\int_{-m}^{+\infty} (f_0^\sharp)' \circ a_\phi(e) a'_\phi(e) b_\phi(e)\, de. \tag{3.11}$$

Now we deal with the case $\widetilde{e}_* = e_* = +\infty$. Since $f^\sharp$ is a nonincreasing function on $\mathbb{R}^+$ and belongs to $L^1(\mathbb{R}^+)$, we deduce that $f^\sharp(s) \to 0$ when $e \to +\infty$. Therefore, according to (3.7), we have $f_0^\sharp \circ a_\phi(e) b_\phi(e) \to 0$ when $e \to +\infty$, and the integration by parts giving (3.11) is also valid in the case $e_* = +\infty$.

Consequently, we have

$$\begin{aligned} \mathcal{J}'(m) &= m + \int_{-m}^{+\infty} (f_0^\sharp)' \circ a_\phi(e) a'_\phi(e) b_\phi(e) de\\ &= m + \int_0^{+\infty} (f_0^\sharp)'(s) b_\phi \circ a_\phi^{-1}(s) ds\\ &= m + \sqrt{m}\int_0^{+\infty} (f_0^\sharp)'(s)\, \beta_1 \circ \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right) ds. \end{aligned}$$

Consider the function

$$h(m, s) = (f_0^\sharp)'(s)\, \beta_1 \circ \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right).$$

Using again Lemma 2.2, we get that h is continuously differentiable with respect to $m \in \mathbb{R}_+^*$ for all $m \in \mathbb{R}_+^* \backslash \{s^2/32\}$, with

$$\frac{\partial h}{\partial m}(m,s) = -\frac{s(f_0^\sharp)'(s)}{2m^{3/2}\,\alpha_1' \circ \alpha_1^{-1}(\frac{s}{\sqrt{m}})}\,\beta_1' \circ \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right).$$

Since $|b_\phi'(e)| \leqslant a_\phi'(e)$, we deduce that

$$\left|\frac{\partial h}{\partial m}(m,s)\right| \lesssim -s(f_0^\sharp)'(s), \quad \text{for all } m \in [m_1, m_2],\ 0 < m_1 < m_2.$$

We now claim that

$$\text{the fonction } s \mapsto s(f_0^\sharp)'(s) \text{ belongs to } L^1(\mathbb{R}_+). \tag{3.12}$$

Indeed, since $f_0^\sharp$ is decreasing, we have

$$\int_0^r s^2 f_0^\sharp(s)ds \geqslant f_0^\sharp(r)\int_0^r s^2 ds = \frac{r^3}{3}f_0^\sharp(r).$$

Hence, using (3.4), we get

$$f_0^\sharp(s) \lesssim \frac{1}{s^3}, \quad \forall s > 0.$$

In particular $sf_0^\sharp(s) \to 0$ when $s \to +\infty$. On the other hand, the function $f_0^\sharp = F \circ a_{\phi_0}^{-1}$ is continuous on $\mathbb{R}_+$, of class $\mathcal{C}^1$ and decreasing on $[0, a_{\phi_0}(e_*))$, vanishing on $[a_{\phi_0}(e_*), +\infty)$ (with possibly $a_{\phi_0}(e_*) = +\infty$). Therefore we can perform the following integration by parts

$$-\int_0^{+\infty} s(f_0^\sharp)'(s)ds = \int_0^{+\infty} f_0^\sharp(s)ds < +\infty.$$

This ends the proof of claim (3.12) and enables to conclude by dominated convergence that $\mathcal{J}'$ is continuously differentiable on $\mathbb{R}_+$ and that

$$\begin{aligned}
\mathcal{J}''(m) &= 1 + \frac{1}{2\sqrt{m}}\int_0^{+\infty} h(m,s)ds + \sqrt{m}\int_0^{+\infty}\frac{\partial h}{\partial m}(m,s)ds \\
&= 1 + \int_0^{+\infty}(f_0^\sharp)'(s)\left(\frac{1}{2\sqrt{m}}\beta_1 \circ \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right) - \frac{s\beta_1' \circ \alpha_1^{-1}\left(\frac{s}{\sqrt{m}}\right)}{2m\,\alpha_1' \circ \alpha_1^{-1}(\frac{s}{\sqrt{m}})}\right)ds \\
&= 1 + \frac{1}{2m}\int_{-m}^{+\infty}\frac{(f_0^\sharp \circ a_\phi)'(e)}{a_\phi'(e)}\left(a_\phi'(e)b_\phi(e) - a_\phi(e)b_\phi'(e)\right)de.
\end{aligned}$$

Finally, observing from (3.9) and from

$$eb_\phi'(e) + 2m\sqrt{2}\int_0^{\theta_m(e)}(\cos\theta)^2(e + m\cos\theta)^{-1/2}\,d\theta = \frac{1}{2}b_\phi(e)$$

that

$$\frac{1}{2ma'_\phi(e)}\left(a'_\phi(e)b_\phi(e)-a_\phi(e)b'_\phi(e)\right)$$

$$=-\frac{(b'_\phi(e))^2}{a'_\phi(e)}+2\sqrt{2}\int_0^{\theta_m(e)}(\cos\theta)^2(e+m\cos\theta)^{-1/2}\,d\theta$$

$$=2\sqrt{2}\int_0^{\theta_m(e)}\left(\cos\theta-\frac{\displaystyle\int_0^{\theta_m(e)}\cos\theta'(e+m\cos\theta')^{-1/2}\,d\theta'}{\displaystyle\int_0^{\theta_m(e)}(e+m\cos\theta')^{-1/2}\,d\theta'}\right)^2(e+m\cos\theta)^{-1/2}\,d\theta,$$

we obtain (3.2) by coming back to the (θ,v) variables. □

4. Control of f

Our previous analysis has allowed the control of the magnetization vector by the relative Hamiltonian and the relative rearrangements. It remains to control the whole distribution function f. To this aim we now write the relative energy in the following form:

$$\mathcal{H}(f)-\mathcal{H}(f_0)=\iint\left(\frac{v^2}{2}+\phi_{f_0}\right)(f-f_0)\,d\theta dv-\frac{1}{2}|M_f-M_{f_0}|^2. \tag{4.1}$$

In particular, this means that the following quantity

$$\iint\left(\frac{v^2}{2}+\phi_{f_0}\right)(f-f_0)\,d\theta dv$$

is controlled and the problem is to show how this quantity controls $f-f_0$. This task was achieved in the context of the gravitational Vlasov-Poisson system [15] using compactness arguments. Here we will rather use a functional inequality established in [14] to get a quantitative control of $\|f-f_0\|_{L^1}$ by this quantity, up to rearrangement terms depending only on f^* and f_0^* which are preserved by the flow. We emphasize that the steady states to Vlasov-Poisson system studied in [15] are compactly supported and this property was essential to successfully drive the stability analysis in this context. Here this assumption is not needed and a much weaker assumption is made in the case of the HMF model. More precisely, we have the following inequality:

Proposition 4.1. *Let f_0 be given by* (1.8) *where F satisfies Assumption 1.1. Then, there exist a constant K_0 depending only on f_0 such that, for all $f\in L^1((1+|v|^2)dvd\theta)$ we have*

$$\begin{aligned}(\|f-f_0\|_{L^1}+\|f_0\|_{L^1}-\|f\|_{L^1})^2\leqslant K_0&\iint\left(\frac{v^2}{2}+\phi_{f_0}\right)(f-f_0)\,d\theta dv\\&+m_0\|f^*-f_0^*\|_{L^1}+\frac{1}{8\pi^2}\int_0^{+\infty}\mu_0(s)^2\beta_{f^*,f_0^*}(s)ds,\end{aligned} \tag{4.2}$$

where $\beta_{f^*,f_0^*}(s)=|\{(\theta,v)\in\mathbb{T}\times\mathbb{R}:f^*(\theta,v)\leqslant s<f_0^*(\theta,v)\}|$, for all $s\geqslant 0$.

Proof. We shall apply Theorem 1 in [14]. We use the rearrangement with respect to $e_0(\theta, v) = \frac{v^2}{2} + \phi_{f_0}$ and recall that the function a_{ϕ_0} is strictly increasing and a one-to-one function from $[\min \phi_0, +\infty)$ to $[0, +\infty)$. Following [14], we introduce the functions

$$B_0(\mu) = \int_0^\mu a_{\phi_0}^{-1}(s)ds, \quad \forall \mu \geqslant 0, \tag{4.3}$$

and

$$H_0(\mu) = \inf_{0<s\leqslant\mu} \frac{B_0(\mu+s) + B_0(\mu-s) - 2B_0(\mu)}{s^2}.$$

Then from Theorem 1 in [14], we have

$$\begin{aligned}(\|f-f_0\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \leqslant & K(f_0) \iint \left(\frac{v^2}{2} + \phi_{f_0}\right)(f-f_0)\, d\theta dv \\ & + \int_0^{+\infty} a_{\phi_0}^{-1}(2\mu_{f_0}(s))\beta_{f^*,f_0^*}(s)ds - \int_0^{+\infty} a_{\phi_0}^{-1}(\mu_{f_0}(s))\beta_{f_0^*,f^*}(s)ds\end{aligned} \tag{4.4}$$

where

$$K(f_0) = 4\int_0^{\|f_0\|_{L^\infty}} \frac{ds}{H_0(\mu_{f_0}(s))}\ , \text{ and} \tag{4.5}$$

$$\beta_{f,g}(s) = \text{meas}\{(\theta, v) \in \mathbb{T}\times\mathbb{R}; f(\theta,v) \leqslant s < g(\theta,v)\}, \quad \forall s \geqslant 0. \tag{4.6}$$

Using the estimates (2.3) we then get from (4.4)

$$\begin{aligned}(\|f-f_0\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \leqslant & K(f_0) \iint \left(\frac{v^2}{2} + \phi_{f_0}\right)(f-f_0)\, d\theta dv \\ & + \frac{1}{8\pi^2}\int_0^{+\infty} \mu_0(s)^2\beta_{f^*,f_0^*}(s)ds + m_0\int_0^{+\infty}\left(\beta_{f_0^*,f^*}(s) + \beta_{f^*,f_0^*}(s)\right)ds\end{aligned} \tag{4.7}$$

Observing that

$$\int_0^{+\infty} \beta_{f,g}(s)ds = \iint (g-f)_+ d\theta dv,$$

we get

$$\int_0^{+\infty}\left(\beta_{f_0^*,f^*}(s) + \beta_{f^*,f_0^*}(s)\right)ds = \|f^* - f_0^*\|_{L^1},$$

and therefore

$$\begin{aligned}(\|f-f_0\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \leqslant & K(f_0) \iint \left(\frac{v^2}{2} + \phi_{f_0}\right)(f-f_0)\, d\theta dv \\ & + \frac{1}{8\pi^2}\int_0^{+\infty} \mu_0(s)^2\beta_{f^*,f_0^*}(s)ds + m_0\|f^* - f_0^*\|_{L^1}.\end{aligned} \tag{4.8}$$

To end the proof of inequality (4.2), it only remains to show that the quantity $K(f_0)$ is finite. First we rewrite $H_0(\mu)$ as

$$\begin{aligned}H_0(\mu) &= \inf_{0<s\leqslant\mu} \frac{B_0(\mu+s) + B_0(\mu-s) - 2B_0(\mu)}{s^2} \\ &= \inf_{0<s\leqslant\mu} \int_0^1 (1-\lambda)((a_{\phi_0}^{-1})'(\mu+\lambda s) + (a_{\phi_0}^{-1})'(\mu-\lambda s))d\lambda.\end{aligned}$$

Then, from the properties of a_ϕ stated in Lemma 2.2, we claim that

$$(a_{\phi_0}^{-1})'(\mu+\lambda s)+(a_{\phi_0}^{-1})'(\mu-\lambda s)\geqslant (a_{\phi_0}^{-1})'(\mu),$$

for all $0\leqslant\lambda\leqslant 1$, $0<s\leqslant\mu$. Indeed, if $\mu\leqslant 16\sqrt{m_0}$ then $\mu-\lambda s\leqslant 16\sqrt{m_0}$, and since the function $a_{\phi_0}^{-1}$ is a concave function on $[0,16\sqrt{m_0}]$, we have $(a_{\phi_0}^{-1})'(\mu-\lambda s)\geqslant (a_{\phi_0}^{-1})'(\mu)$. Therefore we have the desired claim in this case. Similarly, if $\mu\geqslant 16\sqrt{m_0}$ then $\mu+\lambda s\geqslant 16\sqrt{m_0}$, and since the function $a_{\phi_0}^{-1}$ is a convex function on $[16\sqrt{m_0},+\infty]$, we have $(a_{\phi_0}^{-1})'(\mu+\lambda s)\geqslant (a_{\phi_0}^{-1})'(\mu)$. Therefore the above claim holds in both cases. Using this claim, we then get

$$H_0(\mu)\geqslant\frac{1}{2a'_{\phi_0}\circ a_{\phi_0}^{-1}(\mu)},$$

and thus

$$K(f_0)\leqslant 8\int_0^{\|f_0\|_{L^\infty}} a'_{\phi_0}\circ a_{\phi_0}^{-1}(\mu_{f_0}(t))dt.$$

Now we observe that for all $t\geqslant 0$

$$\mu_{f_0}(t)=\left|\left\{F\left(\frac{v^2}{2}+\phi_0(x)\right)>t\right\}\right|=a_{\phi_0}\left(F^{-1}(t)\right),$$

and therefore

$$K(f_0)\leqslant 8\int_0^{\|f_0\|_{L^\infty}} a'_{\phi_0}\left(F^{-1}(t)\right)dt.$$

We then perform the change of variable $e=F^{-1}(t)$ to get

$$K(f_0)\leqslant 8\int_{-m_0}^{e_*} a'_{\phi_0}(e)|F'(e)|de. \tag{4.9}$$

Now we claim that the rhs integral in this inequality is finite. Indeed, assume first that $e_*<+\infty$. The only possible singularities in this integral are at $e=m_0$ and $e=e_*$, since the function $e\mapsto a'_{\phi_0}(e)|F'(e)|$ is continuous on $[-m_0,+\infty)\backslash\{m_0,e_*\}$.

If we suppose that $e^*\neq m_0$, then we have $a'_{\phi_0}(e)|F'(e)|\sim a'_{\phi_0}(e_*)|F'(e)|$ when $e\to e_*$ and, from Lemma 2.2 we have (for $m_0<e_*$ otherwise F vanishes in the neighborhood of m_0) $a'_{\phi_0}(e)|F'(e)|\sim C\log|e-m_0|$ when $e\to m_0$. These two possible singularities are thus integrable (the first is integrable by assumption on F).

If $e^*=m_0$ then our Assumption 1.1 *(i)* ensures that $\int_{-m_0}^{e_*}a'_{\phi_0}(e)|F'(e)|de$ is finite, since $a'_{\phi_0}(e)\sim C\log|e-m_0|$ as $e\to m_0$.

Assume now that $e_*=+\infty$. Using assertion *(iii)* of Lemma 2.2, we have $a'_{\phi_0}(e)|F'(e)|\sim C|F'(e)|/\sqrt{e}$ as $e\to+\infty$, where C is a constant. But, as for (2.12), we have

$$\begin{aligned}\iint F\left(\frac{v^2}{2}+\phi_0(\theta)\right)d\theta dv &= \iint\left(\int_0^{F\left(\frac{v^2}{2}+\phi_0(\theta)\right)}dt\right)d\theta dv\\ &=\int_0^{+\infty}\operatorname{meas}\left\{F\left(\frac{v^2}{2}+\phi_0(\theta)\right)>t\right\}dt\\ &=\int_0^{+\infty}a_{\phi_0}(F^{-1}(t))dt=\int_{-m_0}^{+\infty}a_{\phi_0}(e)|F'(e)|de.\end{aligned}$$

This implies that the integral $\int_{-m_0}^{+\infty} a_{\phi_0}(e)|F'(e)|de$ is convergent. From assertion *(iii)*, we know that $a'_{\phi_0}(e)|F'(e)| \sim C_1|F'(e)|/\sqrt{e} \leqslant C_1|F'(e)|\sqrt{e} \sim C_2 a_{\phi_0}(e)|F'(e)|$ for e large enough, where C_1 and C_2 are some positive constants. This proves the fact that the rhs integral of (4.9) is finite, and ends the proof of Proposition 4.1.

□

5. **Proof of Theorem 1.5**

We first insert identity (4.1) into inequality (4.2) and get

$$(\|f-f_0\|_{L^1}+\|f_0\|_{L^1}-\|f\|_{L^1})^2 \leqslant K_0\left[\mathcal{H}(f)-\mathcal{H}(f_0)+\frac{1}{2}|M_f-M_{f_0}|^2\right] + m_0\|f^*-f_0^*\|_{L^1}+\frac{1}{8\pi^2}\int_0^{+\infty}\mu_0(s)^2\beta_{f^*,f_0^*}(s)ds. \quad (5.1)$$

We write $M_f = |M_f|u(\theta_f)$ where $u(\theta) = (\cos\theta, \sin\theta)^T$, and denote by $f_0(\cdot - \theta_f)$ the fonction $f_0(\cdot-\theta_f)(\theta, v) = f_0(\theta - \theta_f, v)$. We then apply this inequality (5.1) to $f_0(\cdot - \theta_f)$ and get

$$(\|f-f_0(\cdot-\theta_f)\|_{L^1}+\|f_0\|_{L^1}-\|f\|_{L^1})^2 \leqslant K_0\left[\mathcal{H}(f)-\mathcal{H}(f_0)+\frac{1}{2}|M_f-M_{f_0(\cdot-\theta_f)}|^2\right] + m_0\|f^*-f_0^*\|_{L^1}+\frac{1}{8\pi^2}\int_0^{+\infty}\mu_0(s)^2\beta_{f^*,f_0^*}(s)ds.$$

Now we observe that

$$\begin{aligned} M_{f_0(\cdot-\theta_f)} &= \int_0^{2\pi}\rho_{f_0}(\theta-\theta_f)u(\theta)d\theta = \int_0^{2\pi}\rho_{f_0}(\theta)u(\theta+\theta_f)d\theta \\ &= (m_0\cos(\theta_f), m_0\sin(\theta_f))^T = m_0 u(\theta_f). \end{aligned}$$

Therefore

$$(\|f-f_0(\cdot-\theta_f)\|_{L^1}+\|f_0\|_{L^1}-\|f\|_{L^1})^2 \leqslant K_0\left[\mathcal{H}(f)-\mathcal{H}(f_0)+\frac{1}{2}\left(|M_f|-m_0\right)^2\right] + m_0\|f^*-f_0^*\|_{L^1}+\frac{1}{8\pi^2}\int_0^{+\infty}\mu_0(s)^2\beta_{f^*,f_0^*}(s)ds. \quad (5.2)$$

Now we use Corollary 3.2 together with the fact that $\mathcal{J}$ is a C^2 function to conclude that there exist $\delta > 0$ and $C > 0$ such that

$$\mathcal{J}(m)-\mathcal{J}(m_0) \geqslant C(m-m_0)^2 \quad \text{for all } m \in (m_0-\delta, m_0+\delta).$$

Reporting this into estimate (5.2) yields

$$(\|f-f_0(\cdot-\theta_f)\|_{L^1}+\|f_0\|_{L^1}-\|f\|_{L^1})^2 \leqslant K_0\left[\mathcal{H}(f)-\mathcal{H}(f_0)+\frac{1}{2C}\left(\mathcal{J}(|M_f|)-\mathcal{J}(m_0)\right)\right] + m_0\|f^*-f_0^*\|_{L^1}+\frac{1}{8\pi^2}\int_0^{+\infty}\mu_0(s)^2\beta_{f^*,f_0^*}(s)ds.$$

for all f such that $|M_f| \in (m_0-\delta, m_0+\delta)$. Now using inequality (2.8), we get

$$(\|f - f_0(\cdot - \theta_f)\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 \leqslant C\Big[\mathcal{H}(f) - \mathcal{H}(f_0) + C(1+\|f\|_{L^1})\|f^* - f_0^*\|_{L^1} \\ + C\int_0^{+\infty} s^2 (f_0^\sharp(s) - f^\sharp(s))_+ ds + C\int_0^{+\infty} \mu_0(s)^2 \beta_{f^*, f_0^*}(s) ds\Big]. \tag{5.3}$$

for some positive constant C only depending on f_0. To end the proof of Theorem 1.5, we observe that, from the inequality $(a+b)^2 \geqslant \frac{1}{2}a^2 - b^2$,

$$\begin{aligned}(\|f - f_0(\cdot - \theta_f)\|_{L^1} + \|f_0\|_{L^1} - \|f\|_{L^1})^2 &\geqslant \frac{1}{2}\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 - (\|f_0\|_{L^1} - \|f\|_{L^1})^2 \\ &\geqslant \frac{1}{2}\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 - \|f_0^* - f^*\|_{L^1}^2 \\ &\geqslant \frac{1}{2}\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 - (\|f_0\|_{L^1} + \|f\|_{L^1})\,\|f_0^* - f^*\|_{L^1} \\ &\geqslant \frac{1}{2}\|f - f_0(\cdot - \theta_f)\|_{L^1}^2 - \tilde{C}\,(1+\|f\|_{L^1})\,\|f_0^* - f^*\|_{L^1}\end{aligned}$$

with $\tilde{C} = \max(1, \|f_0\|_{L^1})$. We then report this into (5.3) and get inequality (1.12) for all f such that $|M_f| \in (m_0-\delta, m_0+\delta)$.

Let us deduce (1.13) in the case where if f_0 is a compactly supported steady state. In this case, the support of $f_0^\sharp$ is $[0, |\mathrm{Supp} f_0|]$, so

$$\int_0^{+\infty} s^2\left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds \leqslant |\mathrm{Supp} f_0|^2 \int_0^{+\infty} \left(f_0^\sharp(s) - f^\sharp(s)\right)_+ ds \leqslant |\mathrm{Supp} f_0|^2\, \|f^* - f_0^*\|_{L^1}.$$

Furthermore, for all $s \geqslant 0$, we have $\mu_{f_0}(s) \leqslant |\mathrm{Supp} f_0|$, hence

$$\begin{aligned}\int_0^{+\infty} \mu_{f_0}(s)^2 \beta_{f^*, f_0^*}(s) ds \leqslant |\mathrm{Supp} f_0|^2 \int_0^{+\infty} \beta_{f^*, f_0^*}(s) ds &= |\mathrm{Supp} f_0|^2 \iint (f_0^* - f^*)_+ d\theta dv \\ &\leqslant |\mathrm{Supp} f_0|^2\, \|f^* - f_0^*\|_{L^1}\end{aligned}$$

This enables to deduce (1.13) from (1.12) and this ends the proof of Theorem 1.5. ☐

Appendix

Proof of Lemma 2.2. The proof of Item *(i)* is straightforward. Let us prove Item *(ii)*. It is already clear from (2.5) that α_1 is strictly increasing. In order to prove that $\alpha_1'(e)$ is given by (2.6) for all $e \in (-1, 1)$, we perform the change of variable $u = \cos\theta$ in (2.5):

$$\alpha_1(e) = \int_{-1}^1 g(e,u)\, du, \quad \text{where} \quad g(e,u) = 4\sqrt{2}\frac{\sqrt{(e+u)_+}}{\sqrt{1-u^2}}.$$

For $u \in (-1, 1)$, we have

$$0 \leqslant \frac{\partial g}{\partial e} \leqslant q_e(u) = \frac{2\sqrt{2}}{\sqrt{1-e}}\,\frac{1}{\sqrt{(e+u)(1-u)}}\mathbb{1}_{-e\leqslant u \leqslant 1}$$

and, for all $e \in (-1, 1)$,

$$\int_{-1}^1 q_e(u)\, du = \frac{2\sqrt{2}}{\sqrt{1-e}}\pi.$$

Hence, by Brézis-Lieb's Lemma [17], we have $q_e \to q_{e_0}$ in $L^1(-1,1)$, for all $e_0 \in (-1,1)$, and using a generalized dominated convergence theorem as stated in [16] (Appendix A), we deduce that α_1 is $\mathcal{C}^1$ on $(-1,1)$, with

$$\alpha_1'(e) = 2\sqrt{2}\int_{-e}^{1} \frac{du}{\sqrt{(e+u)(1-u^2)}}. \tag{5.4}$$

Performing again the change of variable $u = \cos\theta$ in (5.4) yields (2.6). Now, we perform the change of variable $t = \frac{u+e}{1-u}$ in (5.4) and get, for $e \in (-1,1)$,

$$\alpha_1'(e) = 2\sqrt{2}\int_0^{+\infty} \frac{dt}{\sqrt{t(1+t)(2t+1-e)}}. \tag{5.5}$$

From this expression, we clearly see that α_1' is strictly increasing, which yields the convexity of α_1 on $(-1,1)$. We also deduce that the right-derivative of α_1 at $e=-1$ is finite and its value is given by

$$2\int_0^{+\infty} \frac{dt}{(1+t)\sqrt{t}} = 2\pi.$$

Item *(iii)* is an easy consequence of the following expression, valid for $e > 1$:

$$\alpha_1(e) = 2\sqrt{2}\int_0^{2\pi} \sqrt{e+\cos\theta}\, d\theta.$$

Let us now prove Item *(iv)*. The value $\alpha_1(1) = 16$ is obtained by a direct calculation. In order to prove the equivalent (2.7), we first consider the case $e \to 1$, $e < 1$. The change of variable $s = 1/t$ in (5.5) yields

$$\alpha_1'(e) = 2\sqrt{2}\int_0^{+\infty} \frac{ds}{\sqrt{s(1+s)(2+(1-e)s)}}.$$

Let

$$I_1(e) = 2\sqrt{2}\int_0^{+\infty} \frac{ds}{(1+s)\sqrt{2+(1-e)s}}.$$

From

$$0 \leqslant \frac{1}{\sqrt{s}} - \frac{1}{\sqrt{1+s}} = \frac{1}{\sqrt{s(1+s)}(\sqrt{s}+\sqrt{1+s})} \leqslant \frac{1}{(1+s)\sqrt{s}}, \tag{5.6}$$

we deduce that

$$\left|\alpha_1'(e) - I_1(e)\right| \leqslant 2\sqrt{2}\int_0^{+\infty} \frac{ds}{(1+s)^{3/2}\sqrt{s(2+(1-e)s)}} \leqslant 2\int_0^{+\infty} \frac{ds}{(1+s)^{3/2}\sqrt{s}} = C_0.$$

A direct computation yields

$$I_1(e) = -\frac{2\sqrt{2}}{\sqrt{1+e}}\log\frac{1-e}{(\sqrt{2}+\sqrt{1+e})^2} \sim -2\log(1-e) \quad \text{as} \quad e \to 1,\ e < 1,$$

thus

$$\alpha_1'(e) \sim -2\log(1-e) \quad \text{as} \quad e \to 1,\ e < 1.$$

To deal with the case $e \to 1$, $e > 1$, we perform for $e > 1$ the change of variable $t = \frac{1-\cos\theta}{1+\cos\theta}$ in (2.6):

$$\alpha_1'(e) = 2\sqrt{2}\int_0^{+\infty} \frac{dt}{\sqrt{t(1+t)(t(e-1)+e+1)}}.$$

Using again (5.6), we get

$$\left|\alpha_1'(e) - I_2(e)\right| \leqslant 2\sqrt{2}\int_0^{+\infty} \frac{dt}{(1+t)^{3/2}\sqrt{t(t(e-1)+e+1)}} \leqslant C_0$$

where we used that $t(e-1)+e+1 \geqslant 2$ and where we set

$$I_2(e) = 2\sqrt{2}\int_0^{+\infty} \frac{dt}{(1+t)\sqrt{t(e-1)+e+1}}.$$

Since

$$I_2 = -2\log\frac{e-1}{(\sqrt{2}+\sqrt{1+e})^2} \sim -2\log(e-1) \quad \text{as} \quad e\to 1,\ e>1,$$

we infer that

$$\alpha_1'(e) \sim -2\log(e-1) \quad \text{as} \quad e\to 1,\ e>1,$$

which end the proof of *(iv)*. Finally, Item *(v)* is a straightforward consequence of Items *(i)*, *(ii)*, *(iii)*, *(iv)*, and the proof of Lemma 2.2 is complete. □

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